

INVARIANCE OF THE DRINFELD PAIRING OF A QUANTUM GROUP

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dedicated to Ken-ichi Shinoda in friendship and respect

ABSTRACT. We give two alternative proofs of the invariance of the Drinfeld pairing under the action of the braid group. One uses the Shapovalov form, and the other uses a characterization of the universal R -matrix.

1. INTRODUCTION

Let U be the quantized enveloping algebra over $\mathbb{Q}(q)$ associated to a symmetrizable generalized Cartan matrix $(a_{ij})_{i,j \in I}$. We have the triangular decomposition $U = U^- U^0 U^+$, where U^0 is the Cartan part, $U^+ = \langle e_i \mid i \in I \rangle$ is the positive part, and $U^- = \langle f_i \mid i \in I \rangle$ is the negative part of U respectively. In application of the theory of quantized enveloping algebras to other fields such as mathematical physics and knot theory, the universal R -matrix plays a crucial role. For example, to each representation of the quantized enveloping algebra one can construct a knot invariant by specializing the universal R -matrix. Therefore, it is an important problem to give an explicit description of the universal R -matrix. This problem is equivalent to giving an explicit description of the Drinfeld pairing $\tau : U^+ \times U^- \rightarrow \mathbb{Q}(q)$, which is a bilinear form characterized by certain properties, since the universal R -matrix is defined in terms of τ (see [1]). On the other hand, the Drinfeld pairing τ plays an important role in many aspects of the representation theory. For example, in the finite case various properties of representations when q is not a root of 1 are deduced using properties of τ (see for example [2]).

For $i \in I$, denote by $T_i : U \rightarrow U$ the algebra automorphism introduced by Lusztig [5] (in the finite case there is a different definition due to Levendorskii and Soibelman [4]). It is a lift of the simple reflection of the Weyl group. Let $W = \langle s_i \mid i \in I \rangle$ be the Weyl group. Let $w \in W$, and take a reduced expression $w = s_{i_1} \cdots s_{i_r}$. Set

$$e_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(e_{i_k}), \quad f_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(f_{i_k})$$

2010 *Mathematics Subject Classification.* 20G05, 17B37.

The author was partially supported by Grants-in-Aid for Scientific Research (C) 24540026 from Japan Society for the Promotion of Science.

for $k = 1, \dots, r$. Then there is a well-known formula for the value of

$$(1.1) \quad \tau(e_{\beta_r}^{m_r} \cdots e_{\beta_1}^{m_1}, f_{\beta_r}^{n_r} \cdots f_{\beta_1}^{n_1})$$

(see [3], [4], [5]). In the finite case $\{e_{\beta_r}^{m_r} \cdots e_{\beta_1}^{m_1} \mid m_i \geq 0\}$ (resp. $\{f_{\beta_r}^{n_r} \cdots f_{\beta_1}^{n_1} \mid n_i \geq 0\}$) forms a basis of U^+ (resp. U^-) so that the formula for (1.1) gives an explicit description of τ . A crucial step in the proof of the formula for (1.1) (using Lusztig's definition of T_i) is the following invariance property;

$$(1.2) \quad \tau(T_i^{-1}(x), T_i^{-1}(y)) = \tau(x, y) \\ (x \in U^+ \cap T_i(U^+), y \in U^- \cap T_i(U^-)).$$

The original proof of this result in [5] uses lengthy computation concerning certain generating sets of the algebras $U^\pm \cap T_i(U^\pm)$ (in the detailed account [2] it occupies whole Chapter 8A).

The aim of this note is to give two shorter proofs of (1.2). The first one relies on a relation between the Drinfeld pairing and the Shapovalov form given in Proposition 5.2 below. We think Proposition 5.2 is of independent interest. The second one uses a well known characterization of the universal R -matrix. We hope our investigation in this paper concerning τ including the new proofs of (1.2) will be useful in the future especially in developing the representation theory of quantized enveloping algebras.

The second proof using the universal R -matrix has been obtained in examining the comment by H. Yamane suggesting the possibility of another approach along the line of Levendorskii and Soibelman [4]. I would like to thank Hiroyuki Yamane for this crucial suggestion.

2. QUANTIZED ENVELOPING ALGEBRAS

Assume that we are given a finite-dimensional vector space \mathfrak{h} over \mathbb{Q} and linearly independent subsets $\{h_i\}_{i \in I}$, $\{\alpha_i\}_{i \in I}$ of \mathfrak{h} , \mathfrak{h}^* respectively such that $(\langle \alpha_j, h_i \rangle)_{i,j \in I}$ is a symmetrizable generalized Cartan matrix. Set

$$Q = \sum_{i \in I} \mathbb{Z} \alpha_i, \quad Q^+ = \sum_{i \in I} \mathbb{Z}_{>0} \alpha_i.$$

The Weyl group W is the subgroup of $GL(\mathfrak{h})$ generated by the involutions s_i ($i \in I$) defined by $s_i(h) = h - \langle \alpha_i, h \rangle h_i$ for $h \in \mathfrak{h}$. The contragredient action of W on \mathfrak{h}^* is given by $s_i(\lambda) = \lambda - \langle \lambda, h_i \rangle \alpha_i$ ($i \in I$, $\lambda \in \mathfrak{h}^*$). Set $E = \sum_{i \in I} \mathbb{Q} \alpha_i$. We can take a W -invariant symmetric bilinear form

$$(\ , \) : E \times E \rightarrow \mathbb{Q}$$

such that $\frac{(\alpha_i, \alpha_i)}{2} \in \mathbb{Z}_{>0}$ for any $i \in I$. Then we have $(\alpha_i, \alpha_j) \in \mathbb{Z}$ for $i, j \in I$. We assume that we are given a \mathbb{Z} -form $\mathfrak{h}_{\mathbb{Z}}$ of \mathfrak{h} such that $\langle \alpha_i, \mathfrak{h}_{\mathbb{Z}} \rangle \subset \mathbb{Z}$ and $t_i := \frac{(\alpha_i, \alpha_i)}{2} h_i \in \mathfrak{h}_{\mathbb{Z}}$ for any $i \in I$. For $\gamma = \sum_i n_i \alpha_i \in Q$ set $t_\gamma = \sum_i n_i t_i$. Then we have $\langle \gamma, t_\delta \rangle = (\gamma, \delta)$ for $\gamma, \delta \in Q$.

For $n \in \mathbb{Z}_{\geq 0}$ set

$$[n]_x = \frac{x^n - x^{-n}}{x - x^{-1}} \in \mathbb{Z}[x, x^{-1}], \quad [n]!_x = [n]_x [n-1]_x \cdots [1]_x \in \mathbb{Z}[x, x^{-1}].$$

The quantized enveloping algebra U associated to \mathfrak{h} , $\{h_i\}_{i \in I}$, $\{\alpha_i\}_{i \in I}$, $(\ , \)$, $\mathfrak{h}_{\mathbb{Z}}$ is the associative algebra over $\mathbb{F} = \mathbb{Q}(q)$ generated by the elements k_h , e_i , f_i ($h \in \mathfrak{h}_{\mathbb{Z}}$, $i \in I$) satisfying the relations

$$(2.1) \quad k_0 = 1, \quad k_h k_{h'} = k_{h+h'} \quad (h, h' \in \mathfrak{h}_{\mathbb{Z}}),$$

$$(2.2) \quad k_h e_i k_{-h} = q^{\langle \alpha_i, h \rangle} e_i \quad (h \in \mathfrak{h}_{\mathbb{Z}}, i \in I),$$

$$(2.3) \quad k_h f_i k_{-h} = q^{-\langle \alpha_i, h \rangle} f_i \quad (h \in \mathfrak{h}_{\mathbb{Z}}, i \in I),$$

$$(2.4) \quad e_i f_j - f_j e_i = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}} \quad (i, j \in I),$$

$$(2.5) \quad \sum_{r+s=1-\langle \alpha_j, h_i \rangle} (-1)^r e_i^{(r)} e_j e_i^{(s)} = 0 \quad (i, j \in I, i \neq j),$$

$$(2.6) \quad \sum_{r+s=1-\langle \alpha_j, h_i \rangle} (-1)^r f_i^{(r)} f_j f_i^{(s)} = 0 \quad (i, j \in I, i \neq j),$$

where $k_i = k_{t_i}$, $q_i = q^{(\alpha_i, \alpha_i)/2}$ for $i \in I$, and $e_i^{(r)} = \frac{1}{[r]!_{q_i}} e_i^r$, $f_i^{(r)} = \frac{1}{[r]!_{q_i}} f_i^r$ for $i \in I$, $r \in \mathbb{Z}_{\geq 0}$. For $\gamma \in Q$ we set $k_{\gamma} = k_{t_{\gamma}}$.

The associative algebra U is endowed with a structure of Hopf algebra by

$$(2.7) \quad \Delta(k_h) = k_h \otimes k_h,$$

$$\Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i$$

$$(2.8) \quad \varepsilon(k_h) = 1, \quad \varepsilon(e_i) = \varepsilon(f_i) = 0,$$

$$(2.9) \quad S(k_h) = k_h^{-1}, \quad S(e_i) = -k_i^{-1} e_i, \quad S(f_i) = -f_i k_i$$

for $h \in \mathfrak{h}_{\mathbb{Z}}$, $i \in I$. We will sometimes use Sweedler's notation for the coproduct;

$$\Delta(u) = \sum_{(u)} u_{(0)} \otimes u_{(1)} \quad (u \in U),$$

and the iterated coproduct;

$$\Delta_m(u) = \sum_{(u)_m} u_{(0)} \otimes \cdots \otimes u_{(m)} \quad (u \in U).$$

Define subalgebras U^0 , U^+ , U^- , $U^{\geq 0}$, $U^{\leq 0}$ of U by

$$U^0 = \langle k_h \mid h \in \mathfrak{h}_{\mathbb{Z}} \rangle, \quad U^+ = \langle e_i \mid i \in I \rangle, \quad U^- = \langle f_i \mid i \in I \rangle,$$

$$U^{\geq 0} = \langle k_h, e_i \mid h \in \mathfrak{h}_{\mathbb{Z}}, i \in I \rangle, \quad U^{\leq 0} = \langle k_h, f_i \mid h \in \mathfrak{h}_{\mathbb{Z}}, i \in I \rangle.$$

Then we have

$$U^0 = \bigoplus_{h \in \mathfrak{h}_{\mathbb{Z}}} \mathbb{F} k_h.$$

For $\gamma \in Q$ set

$$U_\gamma = \{u \in U \mid k_h u k_h^{-1} = q^{\langle \gamma, h \rangle} u \ (h \in \mathfrak{h}_\mathbb{Z})\}, \quad U_\gamma^\pm = U_\gamma \cap U^\pm.$$

Then we have

$$U^\pm = \bigoplus_{\gamma \in Q^+} U_{\pm\gamma}^\pm.$$

It is known that the multiplication of U induces isomorphisms

$$\begin{aligned} U &\cong U^+ \otimes U^0 \otimes U^- \cong U^- \otimes U^0 \otimes U^+, \\ U^{\geq 0} &\cong U^+ \otimes U^0 \cong U^0 \otimes U^+, \quad U^{\leq 0} \cong U^- \otimes U^0 \cong U^0 \otimes U^- \end{aligned}$$

of vector spaces.

We define an algebra automorphism

$$(2.10) \quad \Phi : U \otimes U \rightarrow U \otimes U$$

by

$$\Phi(u \otimes u') = q^{-(\gamma, \delta)} u k_{-\delta} \otimes u' k_{-\gamma} \quad (\gamma, \delta \in Q, u \in U_\gamma, u' \in U_\delta).$$

Set

$$P = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \mathfrak{h}_\mathbb{Z} \rangle \subset \mathbb{Z}\}, \quad P^+ = \{\lambda \in P \mid \langle \lambda, h_i \rangle \in \mathbb{Z}_{\geq 0} \ (i \in I)\}.$$

For a (left) U -module V and $\lambda \in P$ we set

$$V_\lambda = \{v \in V \mid k_h v = q^{\langle \lambda, h \rangle} v \ (h \in \mathfrak{h}_\mathbb{Z})\}.$$

A U -module V is said to be integrable if $V = \bigoplus_{\lambda \in P} V_\lambda$ and for any $v \in V$ and $i \in I$ there exists some $N > 0$ such that $e_i^{(n)} v = f_i^{(n)} v = 0$ for $n \geq N$. For $\lambda \in P^+$ define U -modules $V_+(\lambda)$, $V_-(-\lambda)$ by

$$\begin{aligned} V_+(\lambda) &= U / \left(\sum_{h \in \mathfrak{h}_\mathbb{Z}} U(k_h - q^{\langle \lambda, h \rangle}) + \sum_{i \in I} U e_i + \sum_{i \in I} U f_i^{(\langle \lambda, h_i \rangle + 1)} \right), \\ V_-(-\lambda) &= U / \left(\sum_{h \in \mathfrak{h}_\mathbb{Z}} U(k_h - q^{-\langle \lambda, h \rangle}) + \sum_{i \in I} U f_i + \sum_{i \in I} U e_i^{(\langle \lambda, h_i \rangle + 1)} \right). \end{aligned}$$

They are known to be irreducible integrable U -modules. For $\lambda \in P^+$ we set $v_\lambda = \bar{1} \in V_+(\lambda)$, and $v_{-\lambda} = \bar{1} \in V_-(-\lambda)$.

For U -modules V, V' we regard $V \otimes V'$ as a U -module via the co-product $\Delta : U \rightarrow U \otimes U$. If V and V' are integrable, then so is $V \otimes V'$.

The following result follows easily from the proof of [2, Lemma 2.1].

PROPOSITION 2.1. *The following conditions on $u \in U$ are equivalent to each other:*

- (a) $u \in U^{\geq 0}$ (resp. $u \in U^{\leq 0}$),
- (b) for any integrable U -module V and for any $\lambda \in P^+$ we have $u(V \otimes v_\lambda) \subset V \otimes v_\lambda$ (resp. $u(V \otimes v_{-\lambda}) \subset V \otimes v_{-\lambda}$),
- (c) for any integrable U -module V and for any $\lambda \in P^+$ we have $u(v_\lambda \otimes V) \subset v_\lambda \otimes V$ (resp. $u(v_{-\lambda} \otimes V) \subset v_{-\lambda} \otimes V$).

3. BRAID GROUP ACTION

We set

$$\exp_x(y) = \sum_{n=0}^{\infty} \frac{x^{n(n-1)/2}}{[n]!_x} y^n \in (\mathbb{Q}(x))[[y]].$$

Then we have $\exp_x(y) \exp_{x^{-1}}(-y) = 1$.

For $i \in I$ and $t \in \mathbb{F}^\times$ we set

$$\sigma_i(t) = \exp_{q_i}(tq_i^{-1}k_ie_i) \exp_{q_i}(-t^{-1}f_i) \exp_{q_i}(tq_ik_i^{-1}e_i)$$

(see [6]). It is regarded as an invertible operator on a integrable U -module. Moreover, for any integrable U -module V and any $\lambda \in P$ we have $\sigma_i(t)V_\lambda = V_{s_i\lambda}$. If we are give $t_i \in \mathbb{F}^\times$ for each $i \in I$, then the family $\{\sigma_i(t_i)\}_{i \in I}$ satisfies the braid relations. We have

$$\begin{aligned} \sigma_i(t) &= \exp_{q_i}(tq_i^{-n-1}k_i^{n+1}e_i) \exp_{q_i}(-t^{-1}q_i^{-n}k_i^{-n}f_i) \exp_{q_i}(tq_i^{-n+1}k_i^{n-1}e_i) \\ &= \exp_{q_i}(-t^{-1}q_i^{-n-1}k_i^{-n-1}f_i) \exp_{q_i}(tq_i^{-n}k_i^n e_i) \exp_{q_i}(-t^{-1}q_i^{-n+1}k_i^{-n+1}f_i) \end{aligned}$$

for any $n \in \mathbb{Z}$.

For $i \in I$ we define operators $q_i^{\pm h_i(h_i+1)/2}$ and $q_i^{\pm h_i(h_i-1)/2}$ on a integrable U -module V by

$$q_i^{\pm h_i(h_i+1)/2}v = q_i^{\pm \lambda(h_i)(\lambda(h_i)+1)/2}v, \quad q_i^{\pm h_i(h_i-1)/2}v = q_i^{\pm \lambda(h_i)(\lambda(h_i)-1)/2}v$$

for $\lambda \in P$, $v \in V_\lambda$. Then in the notation of [5] we have

$$T'_{i,-1} = q_i^{-h_i(h_i+1)/2} \sigma_i(-1), \quad T''_{i,-1} = q_i^{-h_i(h_i-1)/2} \sigma_i(1),$$

and $T'_{i,1} = (T''_{i,-1})^{-1}$, $T''_{i,1} = (T'_{i,-1})^{-1}$.

REMARK 3.1. If we extend the base field $\mathbb{F} = \mathbb{Q}(q)$ to $\mathbb{Q}(q^{1/4})$, we can write

$$\begin{aligned} \sigma_i(t) &= q_i^{h_i^2/2} \exp_{q_i}(te_i) q_i^{-h_i^2/4} \exp_{q_i}(-t^{-1}f_i) q_i^{-h_i^2/4} \exp_{q_i}(te_i) \\ &= q_i^{h_i^2/2} \exp_{q_i}(-t^{-1}f_i) q_i^{-h_i^2/4} \exp_{q_i}(te_i) q_i^{-h_i^2/4} \exp_{q_i}(-t^{-1}f_i) \\ &= \exp_{q_i}(te_i) q_i^{-h_i^2/4} \exp_{q_i}(-t^{-1}f_i) q_i^{-h_i^2/4} \exp_{q_i}(te_i) q_i^{h_i^2/2} \\ &= \exp_{q_i}(-t^{-1}f_i) q_i^{-h_i^2/4} \exp_{q_i}(te_i) q_i^{-h_i^2/4} \exp_{q_i}(-t^{-1}f_i) q_i^{h_i^2/2}. \end{aligned}$$

In the following we set $T_i = \sigma_i(-1)^{-1} q_i^{h_i(h_i+1)/2}$. In the notation of [5] we have $T_i = T''_{i,1}$. There exists a unique algebra automorphism $T_i : U \rightarrow U$ such that for any integrable U -module V we have $T_i uv = T_i(u)T_i v$ ($u \in U, v \in V$). Then we have $T_i(U_\gamma) = U_{s_i\gamma}$ for $\gamma \in Q$. The

action of T_i on U is given by

$$\begin{aligned} T_i(k_h) &= k_{s_i h}, \quad T_i(e_i) = -f_i k_i, \quad T_i(f_i) = -k_i^{-1} e_i \quad (h \in \mathfrak{h}_{\mathbb{Z}}), \\ T_i(e_j) &= \sum_{r+s=-\langle \alpha_j, h_i \rangle} (-1)^r q_i^{-r} e_i^{(s)} e_j e_i^{(r)} \quad (j \in I, i \neq j), \\ T_i(f_j) &= \sum_{r+s=-\langle \alpha_j, h_i \rangle} (-1)^r q_i^r f_i^{(r)} f_j f_i^{(s)} \quad (j \in I, i \neq j) \end{aligned}$$

(see [5]). We can easily check that

$$(3.1) \quad \Phi \cdot (T_i \otimes T_i) = (T_i \otimes T_i) \cdot \Phi.$$

For $i \in I$ and integrable U -modules V, V' we define operators $Z_i : V \otimes V' \rightarrow V \otimes V'$ and $R_i : V \otimes V' \rightarrow V \otimes V'$ by

$$Z_i = \exp_{q_i}((q_i - q_i^{-1})f_i \otimes e_i), \quad R_i = \exp_{q_i}^{-1}(-(q_i - q_i^{-1})e_i \otimes f_i).$$

They are invertible with

$$Z_i^{-1} = P(R_i),$$

where $P(x \otimes y) = y \otimes x$.

The following result is well-known (see [3], [4], [5]).

PROPOSITION 3.2. *Let V and V' be integrable U -modules. Then as an operator on $V \otimes V'$ we have*

$$T_i = (T_i \otimes T_i) \cdot Z_i = \Phi^{-1}(R_i^{-1}) \cdot (T_i \otimes T_i).$$

LEMMA 3.3. *For $u \in U$ we have*

$$(3.2) \quad \Delta(T_i^{-1}(u)) = Z_i^{-1} \cdot (T_i^{-1} \otimes T_i^{-1})(\Delta(u)) \cdot Z_i,$$

$$(3.3) \quad \Delta(T_i(u)) = \Phi^{-1}(R_i^{-1}) \cdot (T_i \otimes T_i)(\Delta(u)) \cdot \Phi^{-1}(R_i).$$

as operators on the tensor product of two integrable U -modules.

PROOF. By Proposition 3.2 we have

$$\begin{aligned} \Delta(T_i^{-1}(u)) &= T_i^{-1} \cdot \Delta(u) \cdot T_i \\ &= Z_i^{-1} \cdot (T_i^{-1} \otimes T_i^{-1}) \cdot \Delta(u) \cdot (T_i \otimes T_i) \cdot Z_i \\ &= Z_i^{-1} \cdot (T_i^{-1} \otimes T_i^{-1})(\Delta(u)) \cdot Z_i, \end{aligned}$$

and hence (3.2) holds. The proof of (3.3) is similar. \square

Using Proposition 3.2 we see easily the following (see [8, Lemma 2.8]).

LEMMA 3.4. *We have*

$$\begin{aligned} \Delta(T_i(U^{\geq 0})) &\subset U \otimes T_i(U^{\geq 0}), \quad \Delta(T_i(U^{\leq 0})) \subset T_i(U^{\leq 0}) \otimes U, \\ \Delta(T_i^{-1}(U^{\geq 0})) &\subset T_i^{-1}(U^{\geq 0}) \otimes U, \quad \Delta(T_i^{-1}(U^{\leq 0})) \subset U \otimes T_i^{-1}(U^{\leq 0}). \end{aligned}$$

LEMMA 3.5. *We have*

$$\begin{aligned} U^+ \cap T_i(U^{\geq 0}) &= U^+ \cap T_i(U^+), & U^- \cap T_i(U^{\leq 0}) &= U^- \cap T_i(U^-), \\ U^+ \cap T_i^{-1}(U^{\geq 0}) &= U^+ \cap T_i^{-1}(U^+), & U^- \cap T_i^{-1}(U^{\leq 0}) &= U^- \cap T_i^{-1}(U^-). \end{aligned}$$

PROOF. We only show the first formula since the proof of the others are similar.

Let $u \in U^+ \cap T_i(U^{\geq 0})$. Let V be an integrable U -module, and let $v \in V$. For $\lambda \in P^+$ we have

$$T_i^{-1}(u)(v \otimes v_\lambda) = (Z_i^{-1} \cdot (T_i^{-1} \otimes T_i^{-1})(\Delta(u)))(v \otimes v_\lambda)$$

by (3.2). By Lemma 3.4 we have

$$\Delta(u) \in u \otimes 1 + U^{\geq 0} \otimes \left(\bigoplus_{\gamma \in Q^+ \setminus \{0\}} U_\gamma^+ \right) \cap T_i(U^{\geq 0}),$$

and hence

$$(T_i^{-1} \otimes T_i^{-1})(\Delta(u)) \in T_i^{-1}(u) \otimes 1 + U \otimes U^0 \left(\bigoplus_{\gamma \in Q^+ \setminus \{0\}} U_\gamma^+ \right).$$

Therefore, we have

$$T_i^{-1}(u)(v \otimes v_\lambda) = Z_i^{-1}(T_i^{-1}(u)v \otimes v_\lambda) = T_i^{-1}(u)v \otimes v_\lambda.$$

Write

$$T_i^{-1}(u) = \sum_{h \in \mathfrak{h}_\mathbb{Z}} u_h k_h \quad (u_\gamma \in U^+).$$

Then we have

$$\begin{aligned} T_i^{-1}(u)(v \otimes v_\lambda) &= \sum_h u_h (k_h v \otimes k_h v_\lambda) = \sum_h q^{\langle \lambda, h \rangle} u_h (k_h v \otimes v_\lambda) \\ &= \sum_h q^{\langle \lambda, h \rangle} u_h k_h v \otimes v_\lambda, \\ T_i^{-1}(u)v \otimes v_\lambda &= \sum_h u_h k_h v \otimes v_\lambda, \end{aligned}$$

and hence $\sum_{h \in \mathfrak{h}_\mathbb{Z}} (q^{\langle \lambda, h \rangle} - 1) u_h V = \{0\}$ for any integrable U -module V . By [5, 3.5.4] we obtain $\sum_{h \in \mathfrak{h}_\mathbb{Z}} (q^{\langle \lambda, h \rangle} - 1) u_h = 0$ for any $\lambda \in P^+$. From this we see easily that $u_h = 0$ for any $h \neq 0$. \square

By Lemma 3.4 and 3.5 we obtain

$$(3.4) \quad \Delta(U^+ \cap T_i(U^+)) \subset U^{\geq 0} \otimes (U^+ \cap T_i(U^+)),$$

$$(3.5) \quad \Delta(U^- \cap T_i(U^-)) \subset (U^- \cap T_i(U^-)) \otimes U^{\leq 0},$$

$$(3.6) \quad \Delta(U^+ \cap T_i^{-1}(U^+)) \subset (U^+ \cap T_i^{-1}(U^+))U^0 \otimes U^{\geq 0},$$

$$(3.7) \quad \Delta(U^- \cap T_i^{-1}(U^-)) \subset U^{\leq 0} \otimes (U^- \cap T_i^{-1}(U^-))U^0.$$

4. DRINFELD PAIRING

Set

$$\tilde{U}^0 = \bigoplus_{\gamma \in Q} \mathbb{F}k_\gamma \subset U^0, \quad \tilde{U}^{\geq 0} = \tilde{U}^0 U^+, \quad \tilde{U}^{\leq 0} = \tilde{U}^0 U^-.$$

The Drinfeld pairing is the bilinear form

$$\tau : \tilde{U}^{\geq 0} \otimes \tilde{U}^{\leq 0} \rightarrow \mathbb{F}$$

characterized by the following properties:

$$(4.1) \quad \tau(x, y_1 y_2) = (\tau \otimes \tau)(\Delta(x), y_1 \otimes y_2) \quad (x \in \tilde{U}^{\geq 0}, y_1, y_2 \in \tilde{U}^{\leq 0}),$$

$$(4.2) \quad \tau(x_1 x_2, y) = (\tau \otimes \tau)(x_2 \otimes x_1, \Delta(y)) \quad (x_1, x_2 \in \tilde{U}^{\geq 0}, y \in \tilde{U}^{\leq 0}),$$

$$(4.3) \quad \tau(k_\gamma, k_\delta) = q^{-(\gamma, \delta)} \quad (\gamma, \delta \in Q),$$

$$(4.4) \quad \tau(e_i, f_j) = -\delta_{ij}(q_i - q_i^{-1})^{-1} \quad (i, j \in I),$$

$$(4.5) \quad \tau(e_i, k_\gamma) = \tau(k_\gamma, f_i) = 0 \quad (i \in I, \gamma \in Q).$$

It satisfies the following properties:

$$(4.6) \quad \tau(x k_\gamma, y k_\delta) = \tau(x, y) q^{-(\gamma, \delta)} \quad (x \in U^+, y \in U^-, \gamma, \delta \in Q),$$

$$(4.7) \quad \tau(U_\gamma^+, U_{-\delta}^-) = \{0\} \quad (\gamma, \delta \in Q^+, \gamma \neq \delta),$$

$$(4.8) \quad \tau|_{U_\gamma^+ \times U_{-\gamma}^-} \text{ is non-degenerate} \quad (\gamma \in Q^+),$$

$$(4.9) \quad \tau(Sx, Sy) = \tau(x, y) \quad (x \in \tilde{U}^{\geq 0}, y \in \tilde{U}^{\leq 0}).$$

Moreover, for $x \in \tilde{U}^{\geq 0}$, $y \in \tilde{U}^{\leq 0}$ we have

$$(4.10) \quad xy = \sum_{(x)_2, (y)_2} \tau(x_{(0)}, y_{(0)}) \tau(x_{(2)}, Sy_{(2)}) y_{(1)} x_{(1)},$$

$$(4.11) \quad yx = \sum_{(x)_2, (y)_2} \tau(Sx_{(0)}, y_{(0)}) \tau(x_{(2)}, y_{(2)}) x_{(1)} y_{(1)}$$

(see [7]).

For the sake of completeness we include proofs of several well-known facts concerning τ .

LEMMA 4.1 (see Proposition 38.1.6 of [5]). *We have*

$$U^+ \cap T_i(U^+) = \{u \in U^+ \mid \tau(u, U^- f_i) = \{0\}\},$$

$$U^- \cap T_i(U^-) = \{u \in U^- \mid \tau(U^+ e_i, u) = \{0\}\},$$

$$U^+ \cap T_i^{-1}(U^+) = \{u \in U^+ \mid \tau(u, f_i U^-) = \{0\}\},$$

$$U^- \cap T_i^{-1}(U^-) = \{u \in U^- \mid \tau(e_i U^+, u) = \{0\}\}.$$

PROOF. We only show the first formula since the proof of the others are similar.

Assume $u \in U^+ \cap T_i(U^+)$. By $U^+ \cap T_i(U^+) \subset \bigoplus_{\gamma \in Q^+ \cap s_i Q^+} U_\gamma^+$ and (4.7) we have $\tau(U^+ \cap T_i(U^+), f_i) = 0$. Hence by (3.4) we obtain

$$\tau(u, U^- f_i) = \sum_{(u)} \tau(u_{(0)}, U^-) \tau(u_{(1)}, f_i) = \{0\}.$$

Assume $u \in U^+$ satisfies $\tau(u, U^- f_i) = \{0\}$. We have only to show $T_i^{-1}(u) \in U^{\geq 0}$. By Proposition 2.1 it is sufficient to show that for any integrable U -module V and any $\lambda \in P^+$ we have

$$(4.12) \quad T_i^{-1}(u)(V \otimes v_\lambda) \subset V \otimes v_\lambda.$$

We first show

$$(4.13) \quad \Delta(u) \subset \tilde{U}^{\geq 0} \otimes \left(\bigoplus_{\gamma \in Q^+ \setminus \mathbb{Z}_{>0} \alpha_i} U_\gamma^+ \right).$$

For $r > 0$ define $u_r \in U^+$ by

$$\Delta(u) \in \sum_{r>0} u_r k_i^r \otimes e_i^r + \tilde{U}^{\geq 0} \otimes \left(\bigoplus_{\gamma \in Q^+ \setminus \mathbb{Z}_{>0} \alpha_i} U_\gamma^+ \right).$$

Then for $y \in U^-$, $m > 0$ we have

$$\begin{aligned} 0 = \tau(u, y f_i^m) &= \sum_{(u)} \tau(u_{(0)}, y) \tau(u_{(1)}, f_i^m) = \tau(u_m k_i^m, y) \tau(e_i^m, f_i^m) \\ &= \tau(u_m, y) \tau(e_i^m, f_i^m). \end{aligned}$$

By $\tau(e_i^m, f_i^m) \neq 0$ we obtain $u_m = 0$ for any $m > 0$. We have verified (4.13).

On the other hand by $U_\gamma^+ f_i^m \subset \sum_{r=0}^m f_i^r U^0 U_{\gamma-r\alpha_i}^+$ we have

$$(4.14) \quad m \in \mathbb{Z}_{\geq 0}, \gamma \in Q^+ \setminus \mathbb{Z}_{\geq 0} \alpha_i \implies U_\gamma^+ f_i^m v_\lambda = \{0\}.$$

Now we can show (4.12). By (3.2) we have

$$\begin{aligned} T_i^{-1}(u)(V \otimes v_\lambda) &= Z_i^{-1}(T_i^{-1} \otimes T_i^{-1}) \Delta(u)(V \otimes T_i v_\lambda) \\ &= Z_i^{-1}(T_i^{-1} \otimes T_i^{-1}) \Delta(u)(V \otimes f_i^{\langle \lambda, h_i \rangle} v_\lambda). \end{aligned}$$

By (4.13), (4.14) we have

$$\Delta(u)(V \otimes f_i^{\langle \lambda, h_i \rangle} v_\lambda) = (u \otimes 1)(V \otimes f_i^{\langle \lambda, h_i \rangle} v_\lambda) \subset V \otimes f_i^{\langle \lambda, h_i \rangle} v_\lambda,$$

and hence

$$T_i^{-1}(u)(V \otimes v_\lambda) \subset Z_i^{-1}(V \otimes T_i^{-1} f_i^{\langle \lambda, h_i \rangle} v_\lambda) = Z_i^{-1}(V \otimes v_\lambda) = V \otimes v_\lambda.$$

□

LEMMA 4.2 (see Lemma 38.1.2 of [5]). *The multiplication of U induces isomorphisms*

$$\begin{aligned} U^+ &\cong \mathbb{F}[e_i] \otimes (U^+ \cap T_i^{\pm 1}(U^+)) \cong (U^+ \cap T_i^{\pm 1}(U^+)) \otimes \mathbb{F}[e_i], \\ U^- &\cong \mathbb{F}[f_i] \otimes (U^- \cap T_i^{\pm 1}(U^-)) \cong (U^- \cap T_i^{\pm 1}(U^-)) \otimes \mathbb{F}[f_i]. \end{aligned}$$

PROOF. We only show $U^+ \cong \mathbb{F}[e_i] \otimes (U^+ \cap T_i(U^+))$ since other formulas are proved similarly. The injectivity of $\mathbb{F}[e_i] \otimes (U^+ \cap T_i(U^+)) \rightarrow U^+$ follows from $T_i^{-1}(\mathbb{F}[e_i]) \otimes T_i^{-1}(U^+ \cap T_i(U^+)) \subset U^{\leq 0} \otimes U^+ \cong U$. Hence it is sufficient to show that for any $\gamma \in Q$ we have

$$\dim U_\gamma^+ = \sum_{r \geq 0} \dim(U_{\gamma-r\alpha_i}^+ \cap T_i(U^+)).$$

For $\delta \in Q$ we have $\dim(U_\delta^- \cap U^- f_i) = \dim U_{-(\delta-\alpha_i)}^- = \dim U_{\delta-\alpha_i}^+$, and hence $\dim(U_\delta^+ \cap T_i(U^+)) = \dim U_\delta^+ - \dim U_{\delta-\alpha_i}^+$ by Lemma 4.1, (4.7), (4.8). It follows that

$$\sum_{r \geq 0} \dim(U_{\gamma-r\alpha_i} \cap T_i(U^+)) = \sum_{r \geq 0} (\dim U_{\gamma-r\alpha_i}^+ - \dim U_{\gamma-(r+1)\alpha_i}^+) = \dim U_\gamma^+$$

since $\dim U_{\gamma-r\alpha_i}^+ = 0$ for $r \gg 0$. \square

LEMMA 4.3 (see Proposition 38.2.3 of [5]). (i) For $x \in U^+ \cap T_i(U^+)$, $y \in U^- \cap T_i(U^-)$, $m, n \in \mathbb{Z}_{\geq 0}$ we have

$$\tau(xe_i^m, yf_i^n) = \tau(x, y)\tau(e_i^m, f_i^n).$$

(ii) For $x \in U^+ \cap T_i^{-1}(U^+)$, $y \in U^- \cap T_i^{-1}(U^-)$, $m, n \in \mathbb{Z}_{\geq 0}$ we have

$$\tau(e_i^m x, f_i^n y) = \tau(x, y)\tau(e_i^m, f_i^n).$$

PROOF. We only show (i) since (ii) is proved similarly. For $x \in U^+ \cap T_i(U^+)$, $y \in U^- \cap T_i(U^-)$, $m, n \in \mathbb{Z}_{\geq 0}$ we have

$$\begin{aligned} \tau(xe_i^m, yf_i^n) &= (\tau \otimes \tau)(e_i^m \otimes x, \Delta(yf_i^n)) \\ &= (\tau \otimes \tau)(e_i^m \otimes x, \sum_{(y)} y_{(0)} f_i^n \otimes y_{(1)} k_i^{-n}) \\ &= \sum_{(y)} \tau(e_i^m, y_{(0)} f_i^n) \tau(x, y_{(1)} k_i^{-n}) \\ &= \sum_{(y)} (\tau \otimes \tau)(\Delta(e_i^m), y_{(0)} \otimes f_i^n) \tau(x, y_{(1)}) \\ &= \sum_{(y)} \tau(k_i^m, y_{(0)}) \tau(e_i^m, f_i^n) \tau(x, y_{(1)}) = \tau(e_i^m, f_i^n) \tau(x, y). \end{aligned}$$

Here, the second identity follows from Lemma 4.1, and the fifth identity is a consequence of (3.5) and Lemma 4.1. The statement (i) is proved. \square

5. INVARIANCE

5.1. **Main result.** The purpose of this note is to give two simple proofs of the following fact.

THEOREM 5.1 (see Proposition 38.2.1 of [5]). *For $x \in U^+ \cap T_i(U^+)$, $y \in U^- \cap T_i(U^-)$ we have*

$$\tau(T_i^{-1}(x), T_i^{-1}(y)) = \tau(x, y).$$

5.2. The first proof. By the triangular decomposition $U \cong U^- \otimes U^0 \otimes U^+$ we have

$$U = \{(U^- \cap \text{Ker}(\varepsilon))U + U(U^+ \cap \text{Ker}(\varepsilon))\} \oplus U^0.$$

We define a linear map

$$p : U \rightarrow U^0$$

as the projection with respect to this direct sum decomposition. The following fact is crucial.

PROPOSITION 5.2 (see Proposition 19.3.7 of [5]). *Let $\gamma \in Q^+$, and let $x \in U_\gamma^+$, $y \in U_{-\gamma}^-$. Assume*

$$\Delta(x) \in x \otimes 1 + \sum_{\delta \in X} U^{\geq 0} \otimes U_\delta^+$$

for $X \subset Q^+ \setminus \{0\}$. Then we have

$$p(xy) \in k_{-\gamma} \left(\tau(x, y) + \sum_{\delta \in X} \mathbb{F} k_{2\delta} \right).$$

PROOF. Writing

$$\begin{aligned} \Delta(x) &= \sum_r x'_r k_{\delta_r} \otimes x_r \quad (\delta_r \in Q^+, x_r \in U_{\delta_r}^+, x'_r \in U_{\gamma-\delta_r}^+), \\ \Delta(y) &= \sum_s y_s \otimes k_{-\gamma_s} y'_s \quad (\gamma_s \in Q^+, y_s \in U_{-\gamma_s}^-, y'_s \in U_{-(\gamma-\gamma_s)}^-) \end{aligned}$$

we have

$$\begin{aligned} \Delta_2(x) &\in \sum_r x'_r k_{\delta_r} \otimes k_{\delta_r} \otimes x_r + U^{\geq 0} \otimes U^0 (U^+ \cap \text{Ker}(\varepsilon)) \otimes U^+, \\ \Delta_2(y) &\in \sum_s y_s \otimes k_{-\gamma_s} \otimes k_{-\gamma_s} y'_s + U^- \otimes (U^- \cap \text{Ker}(\varepsilon)) U^0 \otimes U^{\leq 0}. \end{aligned}$$

Hence by (4.6), (4.7), (4.10) we have

$$\begin{aligned} p(xy) &= \sum_{\delta_r + \gamma_s = \gamma} \tau(x'_r k_{\delta_r}, y_s) \tau(x_r, S(k_{-\gamma_s} y'_s)) k_{\delta_r - \gamma_s} \\ &= k_{-\gamma} \left(\sum_{\delta_r + \gamma_s = \gamma} \tau(x'_r, y_s) \tau(x_r, S(y'_s)) k_{2\delta_r} \right), \end{aligned}$$

from which we easily obtain our desired result. \square

Now let us give our first proof of Theorem 5.1. We may assume $x \in U_\gamma^+ \cap T_i(U^+)$, $y \in U_{-\gamma}^- \cap T_i(U^-)$ for $\gamma \in Q^+$. By Proposition 5.2 it is sufficient to show

$$p(T_i^{-1}(x)T_i^{-1}(y)) \in k_{-s_i(\gamma)} \left(\tau(x, y) + \sum_{\delta \in Q^+ \setminus \{0\}} \mathbb{F}k_{2\delta} \right).$$

By (3.4), (3.5) we can write

$$\Delta(x) = \sum_r x'_r k_{\delta_r} \otimes x_r, \quad \Delta(y) = \sum_s y_s \otimes k_{-\gamma_s} y'_s,$$

where $\delta_r, \gamma_s \in Q^+ \cap s_i Q^+$, $x_r \in U_{\delta_r}^+ \cap T_i(U^+)$, $x'_r \in U_{\gamma-\delta_r}^+$, $y_s \in U_{-\gamma_s}^- \cap T_i(U^-)$, $y'_s \in U_{-(\gamma-\gamma_s)}^-$. Furthermore, by (3.4), (3.5) and Lemma 4.2 we can write

$$\begin{aligned} \Delta(x_r) &\in \sum_{m \geq 0} e_i^{(m)} k_{\delta_r - m\alpha_i} \otimes x_{rm} + U^{\geq 0}(U^+ \cap T_i(U^+) \cap \text{Ker}(\varepsilon)) \otimes U^+, \\ \Delta(y_s) &\in \sum_{n \geq 0} y_{sn} \otimes k_{-(\gamma_s - n\alpha_i)} f_i^{(n)} + U^- \otimes (U^- \cap T_i(U^-) \cap \text{Ker}(\varepsilon)) U^{\leq 0}, \end{aligned}$$

where $x_{rm} \in U_{\delta_r - m\alpha_i}^+ \cap T_i(U^+)$, $y_{sn} \in U_{-(\gamma_s - n\alpha_i)}^- \cap T_i(U^-)$. Then we have

$$\begin{aligned} \Delta_2(x) - \sum_{r,m} x'_r k_{\delta_r} \otimes e_i^{(m)} k_{\delta_r - m\alpha_i} \otimes x_{rm} \\ \in U^{\geq 0} \otimes U^{\geq 0}(U^+ \cap T_i(U^+) \cap \text{Ker}(\varepsilon)) \otimes U^+, \\ \Delta_2(y) - \sum_{s,n} y_{sn} \otimes k_{-(\gamma_s - n\alpha_i)} f_i^{(n)} \otimes k_{-\gamma_s} y'_s \\ \in U^- \otimes (U^- \cap T_i(U^-) \cap \text{Ker}(\varepsilon)) U^{\leq 0} \otimes U^{\leq 0}. \end{aligned}$$

Hence by (4.6), (4.7), (4.10) we obtain

$$\begin{aligned} (5.1) \quad xy - \sum_{\gamma_s + \delta_r = \gamma - m\alpha_i} \tau(x'_r, y_{sm}) \tau(x_{rm}, S(y'_s)) k_{-(\gamma_s - m\alpha_i)} f_i^{(m)} e_i^{(m)} k_{\delta_r - m\alpha_i} \\ \in (U^- \cap T_i(U^-) \cap \text{Ker}(\varepsilon))U + U(U^+ \cap T_i(U^+) \cap \text{Ker}(\varepsilon)). \end{aligned}$$

In particular, we have

$$p(xy) = \sum_{\gamma_s + \delta_r = \gamma} \tau(x'_r, y_{s0}) \tau(x_{r0}, S(y'_s)) k_{-\gamma + 2\delta_r},$$

and hence

$$(5.2) \quad \tau(x, y) = \sum_{\gamma_s = \gamma, \delta_r = 0} \tau(x'_r, y_{s0}) \tau(x_{r0}, S(y'_s))$$

by Proposition 5.2. Next we apply T_i^{-1} to (5.1). We can easily check that

$$T_i^{-1}(f_i^{(m)} e_i^{(m)}) = e_i^{(m)} f_i^{(m)} \in \begin{bmatrix} k_i \\ m \end{bmatrix} + U(U^+ \cap \text{Ker}(\varepsilon)) + (U^- \cap \text{Ker}(\varepsilon))U,$$

where

$$\begin{bmatrix} k_i \\ m \end{bmatrix} = \prod_{r=1}^m \frac{q_i^{-(r-1)} k_i - q_i^{r-1} k_i^{-1}}{q_i^r - q_i^{-r}}.$$

It follows that

$$\begin{aligned} T_i^{-1}(xy) - \sum_{\gamma_s + \delta_r = \gamma - m\alpha_i} \tau(x'_r, y_{sm}) \tau(x_{rm}, S(y'_s)) \begin{bmatrix} k_i \\ m \end{bmatrix} k_{s_i(\delta_r - \gamma_s)} \\ \in U(U^+ \cap \text{Ker}(\varepsilon)) + (U^- \cap \text{Ker}(\varepsilon))U, \end{aligned}$$

and hence

$$p(T_i^{-1}(xy)) = \sum_{\gamma_s + \delta_r = \gamma - m\alpha_i} \tau(x'_r, y_{sm}) \tau(x_{rm}, S(y'_s)) \begin{bmatrix} k_i \\ m \end{bmatrix} k_{s_i(\delta_r - \gamma_s)}.$$

Note

$$\begin{bmatrix} k_i \\ m \end{bmatrix} \in k_{-m\alpha_i} \left(\mathbb{F}^\times + \sum_{n>0} \mathbb{F} k_{2n\alpha_i} \right).$$

If $\gamma_s + \delta_r = \gamma - m\alpha_i$, then we have

$$s_i(\delta_r - \gamma_s) - m\alpha_i = -s_i\gamma + 2s_i(\delta_r - m\alpha_i).$$

Recall that $x_{rm} \in U_{\delta_r - m\alpha_i}^+ \cap T_i(U^+)$. Hence if $x_{rm} \neq 0$, then $s_i(\delta_r - m\alpha_i) \in Q^+$. Moreover, by $\delta_r \in Q^+ \cap s_i Q^+$, $\delta_r - m\alpha_i = 0$ happens only if $\delta_r = 0$ and $m = 0$. It follows that

$$\begin{aligned} p(T_i^{-1}(xy)) &\in k_{-s_i\gamma} \left(\sum_{\delta_r=0, \gamma_s=\gamma} \tau(x'_r, y_{s0}) \tau(x_{r0}, S(y'_s)) + \sum_{\delta \in Q^+ \setminus \{0\}} \mathbb{F} k_{2\delta} \right) \\ &= k_{-s_i\gamma} \left(\tau(x, y) + \sum_{\delta \in Q^+ \setminus \{0\}} \mathbb{F} k_{2\delta} \right) \end{aligned}$$

by (5.2). The proof is complete.

5.3. The second proof. For each $\gamma \in Q^+$ we denote by $\Theta_\gamma \in U_\gamma^+ \otimes U_{-\gamma}^-$ the canonical element of the non-degenerate bilinear form $\tau|_{U_\gamma^+ \times U_{-\gamma}^-}$. Namely, for bases $\{x_j\}, \{y_j\}$ of $U_\gamma^+, U_{-\gamma}^-$ respectively such that $\tau(x_j, y_k) = \delta_{jk}$ we set $\Theta_\gamma = \sum_j x_j \otimes y_j$. We regard the infinite sum

$$(5.3) \quad \Theta = \sum_{\gamma \in Q^+} \Theta_\gamma$$

as an operator on the tensor product of two integrable U -modules. For $u \in U$ we set

$$\Delta'(u) = P(\Delta(u)),$$

where $P(u_1 \otimes u_2) = u_2 \otimes u_1$. The following fact is crucial.

PROPOSITION 5.3 (see Theorem 4.1.2 of [5]). *We have*

$$(5.4) \quad \Delta'(u) \cdot \Theta = \Theta \cdot (\Phi(\Delta(u))) \quad (u \in U).$$

Moreover, the family $\Theta_\gamma \in U_\gamma^+ \otimes U_{-\gamma}^-$ ($\gamma \in Q^+$) is uniquely determined by the equation (5.4).

Let us give our second proof of Theorem 5.1.

Define a bilinear form

$$\tilde{\tau} : U^+ \times U^- \rightarrow \mathbb{F}$$

by

$$\tilde{\tau}(xe_i^m, yf_i^n) = \tau(T_i^{-1}(x), T_i^{-1}(y))\tau(e_i^m, f_i^n)$$

for $x \in U^+ \cap T_i(U^+)$, $y \in U^- \cap T_i(U^-)$, $m, n \in \mathbb{Z}_{\geq 0}$ (see Lemma 4.2). Then it is sufficient to show $\tau|_{U^+ \times U^-} = \tilde{\tau}$ in view of Lemma 4.3. For $\gamma \in Q^+$ let $\tilde{\Theta}_\gamma$ be the canonical element of $\tilde{\tau}|_{U_\gamma^+ \times U_{-\gamma}^-}$, and set $\tilde{\Theta} = \sum_{\gamma \in Q^+} \tilde{\Theta}_\gamma$. Since $\tau|_{U^+ \times U^-}$ and $\tilde{\tau}$ are uniquely determined by Θ and $\tilde{\Theta}$ respectively, it is sufficient to show $\Theta = \tilde{\Theta}$. Moreover, by the uniqueness in Proposition 5.3 this is equivalent to

$$(5.5) \quad \Delta'(u) \cdot \tilde{\Theta} = \tilde{\Theta} \cdot \Phi(\Delta(u)) \quad (u \in U).$$

For $\gamma \in Q^+ \cap s_i(Q^+)$ let Θ'_γ and Θ''_γ be the canonical elements of $\tau|_{(U_\gamma^+ \cap T_i(U^+)) \times (U_{-\gamma}^- \cap T_i(U^-))}$ and $\tau|_{(U_\gamma^+ \cap T_i^{-1}(U^+)) \times (U_{-\gamma}^- \cap T_i^{-1}(U^-))}$ respectively, and set $\Theta' = \sum_{\gamma \in Q^+ \cap s_i(Q^+)} \Theta'_\gamma$ and $\Theta'' = \sum_{\gamma \in Q^+ \cap s_i(Q^+)} \Theta''_\gamma$. By Lemma 4.3 and the formula

$$\tau(e_i^m, f_i^n) = \delta_{mn} \frac{q_i^{n(n-1)/2}}{(q_i^{-1} - q_i)^n} [n]!_{q_i}$$

we have

$$(5.6) \quad \Theta = \Theta' \cdot R_i = R_i \cdot \Theta'', \quad \tilde{\Theta} = (T_i \otimes T_i)(\Theta'') \cdot R_i.$$

It follows that

$$\begin{aligned}
\Delta'(u) \cdot \tilde{\Theta} &= \Delta'(u) \cdot (T_i \otimes T_i)(\Theta'') \cdot R_i \\
&= (T_i \otimes T_i)((T_i^{-1} \otimes T_i^{-1})(\Delta'(u)) \cdot \Theta'') \cdot R_i \\
&= (T_i \otimes T_i)(R_i^{-1} \cdot \Delta'(T_i^{-1}(u)) \cdot R_i \Theta'') \cdot R_i \\
&= (T_i \otimes T_i)(R_i^{-1} \cdot \Delta'(T_i^{-1}(u)) \cdot \Theta) \cdot R_i \\
&= (T_i \otimes T_i)(R_i^{-1} \Theta \cdot \Phi(\Delta(T_i^{-1}(u)))) \cdot R_i \\
&= (T_i \otimes T_i)(\Theta'' \cdot \Phi(\Delta(T_i^{-1}(u)))) \cdot R_i \\
&= \tilde{\Theta} R_i^{-1} \cdot (T_i \otimes T_i)(\Phi(\Delta(T_i^{-1}(u)))) \cdot R_i \\
&= \tilde{\Theta} R_i^{-1} \cdot \Phi((T_i \otimes T_i)(\Delta(T_i^{-1}(u)))) \cdot R_i \\
&= \tilde{\Theta} \cdot \Phi(\Phi^{-1}(R_i)^{-1} \cdot (T_i \otimes T_i)(\Delta(T_i^{-1}(u))) \cdot \Phi^{-1}(R_i)) \\
&= \tilde{\Theta} \cdot \Phi(\Delta(u))
\end{aligned}$$

by (3.1), (3.2), (3.3), (5.6). We have proved (5.5), and hence our second proof of Theorem 5.1 is complete.

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